Lengths of algebras: introduction

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Semigroup Seminar, 08.10.25

Warm-up

Consider an *algebra* (that is, a linear space equipped with additional multiplication) of $n \times n$ matrices over a given field $\mathbb{F}$. We say that a finite set of matrices $\mathcal{S}$ *generates* it if any element of the algebra can be represented as a linear combination of some products of elements from $\mathcal{S}$.

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How long are the products which we need to generate the algebra? (1, 2, 2n-2 respectively)

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Historical framework

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Theorem (Pappacena, 1997)

Let $\mathcal{A}$ be an associative $\mathbb{F}$-algebra, $m(\mathcal{A})$ the maximal degree of minimal polynomial of its elements and

$$f(d,m) = m\sqrt{\frac{2d}{m-1} + \frac{1}{4}} + \frac{m}{2} - 2.$$

Then $l(\mathcal{A}) < f(\dim \mathcal{A}, m(\mathcal{A}))$.

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This fact requires a more involved proof.

Tools for bounds

Irreducible word

w of length k in $\mathcal{S}$, such that $\forall h : 0 \leq h < k \; w \notin \mathcal{L}_h(\mathcal{S})$

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- Let the elements $m_1, \dots, m_r$ be already defined and the sets $\mathcal{L}_0(\mathcal{S}), \dots, \mathcal{L}_{k-1}(\mathcal{S})$ considered for some $r > 0$, $k > 1$. Denote $s_k = \dim \mathcal{L}_k(\mathcal{S}) - \dim \mathcal{L}_{k-1}(\mathcal{S})$ and define $m_{r+1} = \ldots = m_{r+s_k} = k$.

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- Every m_h is a sum of two previous elements, in particular $m_h \leq 2^{h-1}$.

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A particular direction of the current studies in the area is investigating the classes of *slowly growing length*, that is those which have $\dim \mathcal{A}$ as an upper bound.

Two flavours: computation

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Proposition (Guterman, K., 2017)

For quaternions and octonions viewed as algebras over the field of the reals $\mathbb{R}$ the lengths are 2 and 3.

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Theorem (Guterman, Markova, Khrystik, 2022)

Let G be a finite abelian group, $\mathrm{char}(\mathbb{F}) \nmid |G|$ and $|\mathbb{F}| > |G|$. Then the group algebra $\mathbb{F}G$ is 1-generated and, consequently, $l(\mathbb{F}G) = |G| - 1$.

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Theorem (Guterman, Markova, 2019)

Let S_3 be the symmetric group on 3 elements. Then for any field $\mathbb{F}$ it holds that $l(\mathbb{F}S_3) = 3$.

Semigroups

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A pair $(S, \cdot)$ consisting of a set S and a binary operation $\cdot : S \times S \to S$ such that for any elements $x, y, z \in S$ it holds that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

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Green's relation $\mathcal{R}$

Two elements x, y of a semigroup S are *$\mathcal{R}$-related* if there exist $x', y' \in S$ such that $xx' = y$ and $yy' = x$, or if $x = y$. We denote it by $x\mathcal{R}y$. A semigroup S is called *$\mathcal{R}$-trivial* if for any x, y in S from $x\mathcal{R}y$ it follows that $x = y$.

Rukolaine construction

Let S be an inverse semigroup.

σ-map for idempotents

For an idempotent $e \in S$ consider the finite set $\{e_1, \dots, e_n\}$ of its maximal pre-idempotents. Define $\sigma(e) = e + \sum_{1 \leq < i_1 < \ldots < i_j \leq n} (-1)^j e_{i_1} \ldots e_{i_j}$.

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Properties (Rukolaine, 1984)

Let $a, b \in S$. If $aa^{-1} \neq b^{-1}b$, then $\bar{a}\bar{b} = 0$. If $aa^{-1} = b^{-1}b$, then $\bar{a}\bar{b} = \overline{ab}$. The algebra $\mathcal{A}$ of the finite inverse semigroup S has a basis consisting of elements $\bar{a} = \sigma(aa^{-1})a\sigma(a^{-1}a)$, where $a \in S, a \neq 0$.

Proposition (K., 2024)

Let S be an abelian finite inverse semigroup such that $\mathrm{char}\,\mathbb{F}$ does not divide the order of any subgroup of S and $|\mathbb{F}| > |S|$. Then the (contracted if S contains a zero) semigroup algebra $\mathcal{A}$ of S is 1-generated and, consequently, $l(\mathcal{A}) = |S| - 1$ if S does not contain zero and $l(\mathcal{A}) = |S| - 2$ otherwise.

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We can also consider non-abelian inverse semigroups. Take for example I_n, the inverse semigroup of all partial bijections on n elements. Fix a field $\mathbb{F}$ and let $\mathcal{A}_n$ be the contracted semigroup algebra of I_n over $\mathbb{F}$.

Lengths of inverse semigroup algebras

Proposition (K., 2024)

Let S be an abelian finite inverse semigroup such that $\operatorname{char} \mathbb{F}$ does not divide the order of any subgroup of S and $|\mathbb{F}| > |S|$. Then the (contracted if S contains a zero) semigroup algebra $\mathcal{A}$ of S is 1-generated and, consequently, $l(\mathcal{A}) = |S| - 1$ if S does not contain zero and $l(\mathcal{A}) = |S| - 2$ otherwise.

We can also consider non-abelian inverse semigroups. Take for example I_n, the inverse semigroup of all partial bijections on n elements. Fix a field $\mathbb{F}$ and let $\mathcal{A}_n$ be the contracted semigroup algebra of I_n over $\mathbb{F}$.

Proposition (K., 2024)

The length of the contracted semigroup algebra of I_2 is 3.

Tsetlin libraries T_n

Definition

A a semigroup consisting of non-empty words with no repeating letters in $\{1, \ldots, n\}$ of length n or less with the operation defined as $u \cdot v = (uv)^{\wedge}$, where $u, v \in T_n$, uv is a concatenation of u and v and $^{\wedge} : \{1, \ldots, n\}^* \to \{1, \ldots, n\}^*$ is an operation which sends a given word w to a word $w^{\wedge}$ with all the occurrences of any given letter after the first one removed.

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For example, in T_4 we have $(13) \cdot (234) = (1324)$.
We denote by T_n^1 the semigroup resulting from adjoining an external identity to T_n and we write $C(w)$ for the elements of $1, \ldots, n$ contained in $w \in T_n^1$.

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Remark

For any $n \geq 1$ the semigroups T_n and T_n^1 are $\mathcal{R}$-trivial.

Representation tools

One can view the multiplication by an element $a \in \mathbb{C}T_n$ on the left as a linear operator on $\mathbb{C}T_n^1$ (i.e. consider regular representation of $\mathbb{C}T_n^1$).

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Application of theorem (Steinberg, 2006)

Let $a = \sum\limits_{w \in T_n} \alpha_w w$ be an element of $\mathbb{C}T_n$. Its eigenvalues are given by $\lambda_X = \sum\limits_{w | C(w) \subseteq X} \alpha_w$, where X is a subset of $\{1, \ldots, n\}$.

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Application of propositions (Ayyer, Schilling, Steinberg, Thiéry, 2015)

If these eigenvalues are distinct, the corresponding matrix is diagonalisable. Additionally, for $m \in T_n^1$ it holds that

$$m(a - \lambda_X) = \sum_{w | C(w) \not\subseteq C(m)} \alpha_w mw.$$

Combitnatorial tools

Definition

Let $\mathcal{A}$ be an $\mathbb{F}$-algebra. For $x \in \mathcal{A}$ *minimal multiplicative degree* of x, denoted $m(x)$ is the minimal number m such that $x^m \in \mathcal{L}_{m-1}(\{x\})$ while $x^{m-1} \notin \mathcal{L}_{m-2}(\{x\})$. We also define the *multiplicative degree* of $\mathcal{A}$ to be $m(\mathcal{A}) = \max(m(x), x \in \mathcal{A})$.

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Proposition (K., 2024)

If $l(\mathcal{A}) > m(\mathcal{A})$, then $\dim \mathcal{A} \geq 3l(\mathcal{A}) - 4 + \dim \mathcal{L}_0$.

Results for Tsetlin libraries

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$l(\mathbb{C}T_2) = 3$, $l(\mathbb{C}T_3) = 7$.

Thank you!